This algorithm to solve the inverse problem is simpler than that proposed in [7] which uses the thermal potential. Extraction of the most awkward operations for determining the heat fluxes through the boundary of the initial domain (the true q_2 and effective Q_1) from iteratively finding the boundary location is achieved therein. Not used therein is the linear-fractional transformation of the variable size of the domain into a constant, which results in nonuniformity of the mesh.

However, the method proposed relies on an integral form of the solution of the heat-conduction equation (with Green's functions) and cannot be carried over to the nonlinear case. In the nonlinear case [$\lambda = \lambda(t)$, $c\rho = c(t)\rho(t)$] the algorithm proposed by Alifanov in later papers [8] should be used.

NOTATION

$t(x, \tau)$, true temperature at the point x of the plate at the time τ; $\hat{t}(x_i, \tau)$, temperature measured at the point x_i ($i = 1, 2$) at the time τ; λ, c, ρ, thermal conductivity, specific heat, and density of the plate material; l, plate thickness; x_1, x_2, interior points of the plate; $q_i(\tau)$, heat flux density through the plate surface; $G(x, \xi, \tau, \eta)$, Green's function; T_m, melting point; τ_m, time of the beginning of plate melting; L, specific heat of melting; $s(\tau)$, law of motion of the melted plate surface; q_{1f}, Q_{1f}, densities of the "fictitious" and "effective" fluxes (auxiliary quantities); $\Delta\tau$, difference mesh spacing; N, spacing number.

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DIRECT AND INVERSE HEAT-CONDUCTION PROBLEMS IN A MORE COMPLETE FORMULATION

N. V. Shumakov and A. A. Rostovtseva

UDC 536.2

Direct and inverse heat-conduction problems are formulated and solved for the asymmetric cooling of an infinite plate with nonuniformly distributed and asynchronously acting sources in the case of an inhomogeneous initial distribution.

In electrical engineering, it is important to ensure that powerful electrical motors and generators will conform to the specified thermal operating conditions. In electronics, the construction and use of semiconductor devices ranging from powerful diodes to microcircuits also involves the optimization of temperature conditions of operation.

From a thermophysical point of view, this problem requires the development of experimental and theoretical methods of investigating the temperature in the cooling of a solid with internal sources. In electrical machines the appearance of heat sources is due to Joule-heat losses, remagnetization and eddy currents in magnetic and conducting parts of the machine, friction in the rotating parts, and losses in the circulation of the coolant gas. In semiconductors, heat liberation is due to Joule losses and the Peltier effect. Despite their

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TABLE 1. Inhomogeneity of Temperature Field (Infinite Plate), $\Psi = t_S/t_V = f(\mathrm{Bi}, \mathrm{Fo})$

Bi \ Fo	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2
10	0,89	0,72	0,42	0,28	0,12	1	1
1	0,99	0,96	0,89	0,73	0,63	1	1
10^{-1}	1,0	0,99	0,99	0,97	0,95	1	1
10^{-2}	—	—	—	—	1	1	1
10^{-3}	—	—	—	—	—	1	1
10^{-4}	—	—	—	—	—	—	—

qualitative differences, sources in machines and in semiconductors have in common that their effect depends on the coordinate and on time.

This problem will be considered in the context of electronics. It may be divided into two parts: 1) the problem of calculating the temperature field of a semiconductor device; 2) the problem of experimental investigation of the thermal conditions in the device.

The calculation of the temperature field must be formulated as a direct heat-conduction problem, taking into account the internal heat sources and the inhomogeneous initial distribution. The problem is particularly nonlinear since semiconductor materials have strongly temperature-dependent physical parameters. Examination of the temperature dependence of the heat conduction for germanium and silicon [1] leads to a practically useful conclusion: It is easier to ensure stable operation of semiconductor devices at temperatures above room temperature than at lower temperatures; the stability should be higher in germanium than in silicon.

The basis for a priori specification of boundary conditions is usually inadequate. Their formulation requires a great deal of experimental work on the time dependence.

Consider the formation of a nonsteady temperature field in a solid. Table 1 gives values of $\Psi = t_S/t_V$ (t_S and t_V are the mean-integral temperatures over the surface and the volume, respectively) as a function of $\mathrm{Bi} = \alpha R/\lambda$ and $\mathrm{Fo} = a\tau/R^2$. The values correspond to an infinite plate of thickness 2R. Bi is determined as the ratio of the internal (R/λ) and external $(1/\alpha)$ heat resistances. If the linear dimension of the semiconductor device is R = 0.1-10 cm, the minimum value of Bi is 10^{-4} (cooling in still air — natural convection). For forced convection in air $\mathrm{Bi} = 10^{-3}$-10^{-1}. By using other means of cooling Bi may be raised to 10. Heat calculations involving the solution of the heat-conduction equation are usually made for times such that $\mathrm{Fo} \geq 10^{-4}$ (for finite dimensions). It is evident from Table 1 that when $\mathrm{Bi} \geq 0.1$ the temperature field is significantly inhomogeneous. For bodies with $\mathrm{Bi} < 0.1$ the temperature field is sufficiently homogeneous when $\mathrm{Fo} > 1$. The homogeneity of the field at lower values of Fo remains an open question: The literature has no information on detailed investigations and the existing nomograms for temperature-field calculations [2-4] do not cover this region (the lower left-hand corner of Table 1).

It is evident that for very small times the inhomogeneity of the field is also important at small Bi, especially for bodies with internal heat liberation. However, it remains open to question whether the field can be described by means of the series used at present in solutions of the heat-conduction equation, while it is the region of low Fo that is of particular interest for microelectronics. Calculation of the temperature field according to Table 1 (taking into account the properties of the materials) is possible for heat-liberation pulses of duration $\Delta\tau \geq 10^{-5}$ sec in small instruments ($R \simeq 1$ mm) but only for $\Delta\tau > 0.1$ sec in larger bodies ($R \simeq 10$ cm). For fast-acting circuits, inhomogeneity cannot be taken into account without additional investigations. On the other hand, since the strong temperature dependence of the physical parameters of the materials employed affects the reliability of operation of the device, it is necessary to acknowledge that the quasisteady approach [5] using the electrothermal analogy scarcely reflects a true picture of the situation.

Using the results outlined above and taking into account that the physical properties of semiconductors depend on the coordinate and the temperature, it is clear that the calculation of temperature fields in semiconductor devices is complicated by the need to solve the nonlinear heat-conduction problem in the presence of discrete and distributed sources in inhomogeneous bodies. The solution is of particular interest for finite bodies in a range several orders of magnitude less than $\mathrm{Fo} = 10^{-3}$.

Now consider the temperature field of a semiconductor device in a one-dimensional approximation. The device may be represented in the form of an infinite plate with asymmetric cooling $q_1(\tau) \neq q_2(\tau)$. The plate contains heat sources of various outputs. Separating these into uniformly distributed synchronously acting sources with output per unit volume $w_0(\tau)$ and z independently acting discrete sources, with output per unit volume $w_i(x, \tau)$ $(1 \le i \le z)$ and a heat-liberation region of width δ_i centered on the point x_i. At this stage, it is assumed that the material of the body is homogeneous and isotropic and that its physical properties are independent of temperature. The solution of this linear problem may subsequently be used to find the algorithm for the nonlinear problem

$$\frac{\partial t(x, \tau)}{\partial \tau} = a\frac{\partial^2 t(x, \tau)}{\partial x^2} + \frac{1}{C\rho} w(x, \tau); \tag{1}$$

$$0 \leqslant x \leqslant R, \quad 0 \leqslant \tau < \infty; \tag{2}$$

$$t(x, 0) = A + B\left(\frac{x}{R}\right) + C\left(\frac{x}{R}\right)^2; \tag{3}$$

$$\frac{\partial t(0, \tau)}{\partial x} = -\frac{1}{\lambda} q_1(\tau), \quad \frac{\partial t(R, \tau)}{\partial x} = -\frac{1}{\lambda} q_2(\tau); \tag{4}$$

$$w(x, \tau) = w_0(\tau) + \sum_{i=1}^{z} w_i(x, \tau); \tag{5}$$

$$w_i(x, \tau) = \begin{cases} w_i(\tau), & x \in \left(x_i \pm \dfrac{\delta_i}{2}\right), \\ 0, & x \notin \left(x_i \pm \dfrac{\delta_i}{2}\right). \end{cases} \tag{6}$$

The problem is solved by the method of successive intervals, using the results of [6] and the superposition principle. The solution may be written in the following form

$$\begin{aligned} t(y, n\omega) &= t(y, 0) + \omega\frac{R^2}{\lambda}\sum_{k=0}^{n-1} w_{0,k+1} - (B + 2C) F[1 - y, n\omega] \\ &+ BF[y, n\omega] + 2Cn\omega - \frac{R}{\lambda}\left\{\sum_{k=0}^{n-1} q_{1,k+1}\Delta F[1 - y, (n-k)\omega]\right. \\ &\left.+ \sum_{k=0}^{n-1} q_{2,k+1}\Delta F[y, (n-k)\omega]\right\} + \frac{R^2}{\lambda}\sum_{i=1}^{z}\sum_{k=0}^{n-1} w_{i,k+1}\Delta G_i^v[y, (n-k)\omega], \end{aligned} \tag{7}$$

where

$$y = \frac{x}{R}, \quad \omega = \frac{a\Delta\tau}{R^2}, \quad n = \frac{\tau}{\Delta\tau}, \quad y_i = \frac{x_i}{R};$$

for large times (Fo > 0.1)

$$\begin{aligned} F[y, n\omega] &= n\omega + \frac{1}{3} - y + \frac{y^2}{2} - 2\sum_{m=1}^{\infty}\frac{1}{\mu_m^2}\cos(\mu_m y)\exp(-\mu_m^2 n\omega), \\ \Delta F[y, (n-k)\omega] &= F[y, (n-k)\omega] - F[y, (n-k-1)\omega], \end{aligned} \tag{8}$$

$$\begin{aligned} \Delta G_i^v[y, (n-k)\omega] &= \frac{\delta_i\omega}{R} + 4\sum_{m=1}^{\infty}\frac{1}{\mu_m^3}\cos(\mu_m y_i)\cos(\mu_m y) \\ &\times \sin\left(\frac{\delta_i\mu_m}{2R}\right)\{\exp[-\mu_m^2(n-k-1)\omega] - \exp[-\mu_m^2(n-k)\omega]\}, \end{aligned} \tag{9}$$

the coefficients of the expansions $q_{1,k+1}$, $q_{2,k+1}$, $w_{0,k+1}$, $w_{i,k+1}$ being determined from the formula

$$u_{k+1} = \frac{1}{\Delta\tau} \int_{k\Delta\tau}^{(k+1)\Delta\tau} u(t)\,dt.$$

If the heat-liberation region is narrow ($\delta_i \ll R$), the function describing the action of surface sources is as follows:

$$R\Delta G_i^S [y,\ (n-k)\omega] = \omega + 2 \sum_{m=1}^{\infty} \frac{1}{\mu_m^2} \cos(\mu_m y)\cos(\mu_m y_i)$$
$$\times \{\exp[-\mu_m^2 (n-k-1)\omega] - \exp[-\mu_m^2 (n-k)\omega]\}, \tag{10}$$

and the term $w_{i,k+1}$ in Eq. (7) has the dimensions W/m^2 and is the output per unit surface.

For small time (Fo < 0.1) the functions $F(y, n\omega)$ and $\Delta G_i^V[y, (n-k)\omega]$, $\Delta G_i^S[y, (n-k)\omega]$ take the form

$$F(y, n\omega) = 2\sqrt{n\omega} \sum_{m=1}^{\infty} \left\{ \text{ierfc}\left[\frac{2m+y}{2\sqrt{n\omega}}\right] + \text{ierfc}\left[\frac{y-2m+2}{2\sqrt{n\omega}}\right]\right\}, \tag{11}$$

$$\Delta G_i^V[y,\ (n-k)\omega] = 2\omega(n-k) \sum_{m=-\infty}^{\infty} \left\{ i^2\text{erfc}\left[\frac{y-y_i-\gamma_i+2m}{2\sqrt{(n-k)\omega}}\right]\right.$$
$$- i^2\text{erfc}\left[\frac{y-y_i+\gamma_i+2m}{2\sqrt{(n-k)\omega}}\right] + i^2\text{erfc}\left[\frac{y+y_i-\gamma_i+2m}{2\sqrt{(n-k)\omega}}\right]$$
$$\left. - i^2\text{erfc}\left[\frac{y+y_i+\gamma_i+2m}{2\sqrt{\omega(n-k)}}\right]\right\} - 2\omega(n-k-1)$$
$$\times \sum_{m=-\infty}^{\infty} \left\{ i^2\text{erfc}\left[\frac{y-y_i-\gamma_i+2m}{2\sqrt{\omega(n-k-1)}}\right] - i^2\text{erfc}\left[\frac{y-y_i+\gamma_i+2m}{2\sqrt{\omega(n-k-1)}}\right]\right.$$
$$\left. + i^2\text{erfc}\left[\frac{y+y_i-\gamma_i+2m}{2\sqrt{\omega(n-k-1)}}\right] - i^2\text{erfc}\left[\frac{y+y_i+\gamma_i+2m}{2\sqrt{\omega(n-k-1)}}\right]\right\};\ \gamma_i = \frac{\delta_i}{2R}; \tag{12}$$

$$R\Delta G_i^S[y,\ (n-k)\omega] = \sqrt{\omega(n-k)} \sum_{m=-\infty}^{\infty} \left\{ \text{ierfc}\left[\frac{y-y_i+2m}{2\sqrt{\omega(n-k)}}\right] + \text{ierfc}\left[\frac{y+y_i+2m}{2\sqrt{\omega(n-k)}}\right]\right\}$$

$$-\sqrt{\omega(n-k-1)} \sum_{m=-\infty}^{\infty} \left\{ \text{ierfc}\left[\frac{y-y_i+2m}{2\sqrt{\omega(n-k-1)}}\right] + \text{ierfc}\left[\frac{y+y_i+2m}{2\sqrt{\omega(n-k-1)}}\right]\right\}. \tag{13}$$

Usually, to determine the conditions at the boundary between the solid and the coolant medium, a special experiment is carried out to find the velocity and temperature of the coolant gas, the heat-transfer coefficient being found for steady processes. By using inverse problems to determine the time variation of the boundary conditions for nonsteady heat transfer it is possible to establish both the nonsteady behavior due to the temperature variations of the body itself and that due to variation in the state of the coolant gas. In other words, it is possible to investigate the heat transfer in both steady and transient conditions of motion of the medium.

In the general case the inverse properties formulated on the basis of Eq. (7) has (3z + 3) unknowns; q_1, q_2, w_0, w_i, x_i, δ_i. Writing Eq. (7) for (3z + 3) points leads to a system of equations which, in principle, may be solved. The solution has been obtained for z = 1 (as it is cumbersome, it is not given here). If all (3z + 3) values are to be determined, numerical solution is necessary; if x_i and δ_i are known, the remaining (z + 3) values may be determined analytically.

There is wide scope for the use of the inverse problem in studying the operation of electrical machines: Their size and shape is such as to accommodate, in principle, the required number of pickups, which should be no less than the number of unknowns in the problem.

Current techniques of temperature measurement and semiconductor construction impose definite and important restrictions on experimental work. Since in reality only the end surfaces are accessible for measurements, it is found that the number of unknowns in the problem exceeds the number of points for temperature measurement even for the simplest case z = 1. This case applies to the diode.

To reduce the number of unknowns, the experiment may be set up in such a way that $q_1(\tau) = q_2(\tau) \simeq 0$. Further, for a sufficiently thick crystal (~ 1 mm), it is possible to disregard the thickness of the heat-liberation region δ_i and to regard the discrete source as plane. The remaining three unknowns — the output per unit volume of the distributed sources $w_0(\tau)$, the output per unit volume of the discrete source $w_i(\tau)$, and its position x_i — may be determined by recording the surface-temperature variation of the crystal with time and the variation with time of the total losses in the crystal. A number of experimental procedures are possible. The next problem is to determine the thickness δ_i of the region in which the heat source acts. Another possibility to be investigated is the use of the method outlined in combination with the use of heat-sensitive parameters [7] requires further analysis.

The method of successive intervals may also be used to obtain an approximate three-dimensional picture. This involves the use of a diode with a cellular base. The crystal may then be considered as a collection of independent current tubes, and the temperature variation at several points of the crystal end surfaces may be determined experimentally. The use of the inverse problem allows the transverse distribution of the output per unit volume of the discrete sources in the crystal to be approximately determined.

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SOLUTION OF CONJUGATE PROBLEM IN SUCCESSIVE INTERVALS

L. D. Kalinnikov and N. V. Shumakov

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The conjugate problem is solved for nonsteady heat transfer through a plane wall with convective heat transfer at the edges.

On the basis of experimental data on the startup conditions of heat transfer [1, 2], the energy equation for the thermal boundary layer and the heat-conduction equation for the wall are solved jointly. Having obtained the solution, the variation with time in the temperature of the heat-transfer surfaces and the heat flows in the course of nonsteady heat transfer may be determined for given parameters.

In formulating the problem it is assumed that the flow of liquid is stable, the flow rate is given, and its mean velocity over the cross section is known. The liquid-flow temperature is assumed to be constant and equal to the liquid temperature at the inlet to the heat-transfer section. The liquid is incompressible with constant thermophysical properties. Energy dissipation due to viscosity and heat conduction of the wall material in the longitudinal direction of liquid flow is neglected. The mean heat-transfer coefficient is referred to the difference between the temperature of the heat-transfer surface of the wall and the liquid temperature.

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